ON A DECOMPOSITION THEOREM FOR MEASURES IN EUCLIDEAN n-SPACE(1)

BY

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Introduction. It is the purpose of this paper to extend a decomposition theorem of Mickle [1] (square brackets refer to the bibliography at the end of this paper) for $(n-1)$-dimensional measures in Euclidean n-space R^n, to k-dimensional measures in R^n, for $0<k<n$, k an integer. In this paper we define a measure μ_n^k on the family $\mathcal{B}$ of Borel sets of R^n that satisfies the following conditions:

(a) If $B\in\mathcal{B}$ and $\mu_n^k(B)<+\infty$, then $B=B_1\cup B_2$, where B_1 is countably k-rectifiable (see 4.1) and $\mu_n^k(B_2)=0$.

(b) If $B\in\mathcal{B}$, then

$$F_n^k(B) \leqq \mu_n^k(B) \leqq H_n^k(B), \tag{1}$$

where F_n^k is the Favard k-dimensional measure in R^n (see 7.3) and H_n^k is the Hausdorff k-measure in R^n (see [2] 2.18). Furthermore (see 7.3),

$$F_n^k(B) = \mu_n^k(B), \tag{2}$$

whenever $\mu_n^k(B)<+\infty$ and

$$F_n^k(B) = \mu_n^k(B) = H_n^k(B), \tag{3}$$

whenever B is countably k-rectifiable. Whether (2) holds for every $B\in\mathcal{B}$ is an open question.

(c) μ_n^k is the smallest measure on $\mathcal{B}$ which satisfies a weak projection inequality in the following sense: For almost every R^k(in a sense given in 6.4), the Lebesgue k-dimensional measure of the projection of a Borel set B into R^k is less than or equal to $\mu_n^k(B)$.

(d) If m is a positive integer such that $k<m<n$, and B is a Borel set in R^m, then (see 7.4)

$$\mu_m^k(B) = \mu_n^k(B).$$

While our results are stated in terms of measures on Borel sets, it will be convenient to work with Borel regular Carathéodory outer measures (see

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1.11). The procedures and proofs will follow closely those of Mickle [1] and Federer [2]. However, it should be noted that in the work of Mickle [1] the geometrical arguments needed do not necessitate the use of the group of orthogonal transformations of R^n onto R^n that is used in this paper.

I. **Preliminary considerations.**

1.1. Let R^n denote Euclidean n-space, $\mathcal{B}$ the Borel sets of R^n, G_n the group of orthogonal $n\times n$ matrices with real entries, and σ_n the Haar measure on G_n. Likewise let G_k denote the group of orthogonal $k\times k$ matrices, and σ_k the Haar measure in G_k. For g, an element of G_n, let g_i be the ith row vector of g. That is,

$$g_i = (g_{i1}, g_{i2}, \cdots, g_{in}),$$

where g_{ij} is the entry of g in the ith row and jth column. Let I^k be that $n\times n$ matrix formed from the identity of G_n by setting the last $n-k$ diagonal elements equal to 0. We shall use I^k both for the above matrix and for the mapping effected by the matrix, that is, for the projection of R^n onto the space spanned by the first k basis vectors of R^n. Thus, although I^k considered as a matrix has no inverse, we shall use $(I^k)^{-1}$ for the inverse of the projection mapping. For convenience, we shall set $R^k = I^k(R^n)$. Also let $I_{n-k} = I - I^k$, and $R_{n-k} = I_{n-k}(R^n)$. If y is an element of R^n, and g is an element of G_n, then the length of $I^k g(y)$ and $I_{n-k}g(y)$ are given by

$$| I^k g(y) | = \sum_{i=1}^{k} [(y\cdot g_i)^2]^{1/2},$$

$$| I_{n-k} g(y) | = \sum_{i=k+1}^{n} [(y\cdot g_i)^2]^{1/2},$$

where "$\cdot$" denotes inner product.

1.2. Let S be an element of G_{k+1}. Then S is a $k+1$ by $k+1$ orthogonal matrix. We form an $n\times n$ orthogonal matrix S' from S as follows:

$$S'_i = (S_{i1}, S_{i2}, \cdots, S_{i,k+1}, 0, \cdots, 0) \qquad \text{for } 1 \leqq i \leqq k+1,$$

$$S'_i = (\delta_{i1}, \delta_{i2}, \cdots, \delta_{in}) \qquad \text{for } k+2 \leqq i \leqq n.$$

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

S' may be considered as a continuous function of S.

1.3. Lemma. *For S in G_{k+1} and y in R^n,*

$$S' I^{k+1}(y) = I^{k+1} S'(y).$$

Proof. Let

$$u = (u_1, \cdots, u_n) = S'I^{k+1}(y),$$
$$v = (v_1, \cdots, v_n) = I^{k+1}S'(y),$$
$$I^{k+1}(y) = (y_1, \cdots, y_{k+1}, 0, \cdots, 0).$$
$$u_i = \sum_{p=1}^{n} S'_{ip}(I^{k+1}(y))_p = \sum_{p=1}^{k+1} S'_{ip}y_p,$$
$$v_i = \begin{cases} \sum_{p=1}^{n} S'_{ip}y_p & \text{if } 1 \leqq i \leqq k+1, \\ 0 & \text{if } k+2 \leqq i \leqq n. \end{cases}$$

But if $1 \leqq i \leqq k+1$ and $k+2 \leqq p \leqq n$, then $S'_{ip}=0$; so

$$v_i = \sum_{p=1}^{k+1} S'_{ip}y_p = u_i \qquad \text{if } 1 \leqq i \leqq k+1.$$

Also if $k+2 \leqq i \leqq n$ and $1 \leqq p \leqq k+1$, then $S'_{ip}=0$; so

$$u_i = \sum_{p=1}^{k+1} S'_{ip}y_p = 0 = v_i \qquad \text{if } k+2 \leqq i \leqq n.$$

Hence $u=v$.

1.4. LEMMA. *For S in G_{k+1} and y in R^n*

$$|I^{k+1}S'(y)| = |I^{k+1}(y)|.$$

Proof. S' is in G_n and hence S' preserves length. Therefore by 1.3,

$$|I^{k+1}S'(y)| = |S'I^{k+1}(y)| = |I^{k+1}(y)|.$$

1.5. LEMMA. *For S in G_{k+1} and $\alpha > 0$ let*

$$\mathcal{U}(S, \alpha) = \{x \mid x \in R^n, |I^kS'(x)| < \alpha[S'(x)]k + 1\},$$
$$\mathcal{V}(S, \alpha) = \{x \mid x \in R^n, |I^{k+1}(x)| < (\alpha^2 + 1)^{1/2}[S'(x)]k + 1\}.$$

Then $\mathcal{U}(S, \alpha) = \mathcal{V}(S, \alpha)$.

Proof. First note that for x in either set

$$[S'(x)]_{k+1} = (S'_{k+1} \cdot x) > 0.$$

Now

$$x \in \mathcal{U}(S, \alpha)$$

if and only if

$$|I^kS(x)|^2 = \sum_{i=1}^{k} (S'_i \cdot x)^2 < \alpha^2[(S'(x))_{k+1}]^2$$

if and only if

$$|I^{k+1}S'(x)|^2 = \sum_{i=1}^{k+1} (S_i' \cdot x)^2 < (\alpha^2 + 1)[(S'(x))_{k+1}]^2.$$

By 1.4 then

$$x \in \mathfrak{U}(S, \alpha)$$

if and only if

$$|I^{k+1}(x)|^2 < (\alpha^2 + 1)[(S'(x))_{k+1}]^2$$

if and only if

$$x \in \mathfrak{V}(S, \alpha).$$

1.6. For x a real number, $x > 0$, let

$$y = \beta(x) = \{2[1 - (x^2 + 1)^{-1/2}]\}^{1/2}.$$

Then $\lim_{x\to 0} \beta(x) = 0$ and

$$x = \beta^{-1}(y) = \frac{y(4 - y^2)^{1/2}}{2 - y^2}, \qquad \lim_{y\to 0} \beta^{-1}(y) = 0,$$

and

$$\lim_{x\to 0} \frac{\beta(x)}{x} = 1.$$

1.7. Let $C^{k+1} = \{y \mid y \in R^{k+1}, |y| = 1\}$. For y in C^{k+1}, $0 < \eta < \infty$, let

$$C^0(y, \eta) = \{Z \mid Z \in C^{k+1}, |Z - y| < \eta\},$$
$$C(y, \eta) = \{Z \mid Z \in C^{k+1}, |Z - y| \leqq \eta\},$$
$$D = \{x \mid x \in R^n, |I^{k+1}(x)| > 0\}.$$

D is open in R^n.

Let f be the mapping with domain D and range C^{k+1} given by

$$f(x) = \frac{I^{k+1}(x)}{|I^{k+1}(x)|}.$$

f is continuous onto C^{k+1}.

1.8. LEMMA. *For S in G_{k+1}, $\alpha > 0$,*

$$f^{-1}C^0[S_{k+1}, \beta(\alpha)] = V(S, \alpha).$$

Proof.

$$f^{-1}C^0[S_{k+1}, \beta(\alpha)] \subset D.$$

Also, for x in $\mathfrak{V}(S, \alpha)$, $[S'(x)]_{k+1} > 0$, which implies that $|I^{k+1}S'(x)| > 0$, which by 1.4 implies that $|I^{k+1}(x)|>0$. Hence for any x under consideration, $|I^{k+1}(x)|>0$. Now by 1.3

$$\begin{aligned}[S'(x)]_{k+1} &= [I^{k+1}S'(x)]_{k+1}\\ &= [S'I^{k+1}(x)]_{k+1} = S'_{k+1}\cdot[I^{k+1}(x)]\\ &= S_{k+1}\cdot[I^{k+1}(x)] = (S_{k+1}\cdot f(x))(|I^{k+1}(x)|).\end{aligned}$$

Also

$$|S_{k+1}| = |f(x)| = 1.$$

So

$$|f(x) - S_{k+1}|^2 = 2[1 - S_{k+1}\cdot f(x)].$$

Thus x is an element of $\mathcal{V}(S, \alpha)$ if and only if

$$|I^{k+1}(x)| < (\alpha^2 + 1)^{1/2}[S'(x)]_{k+1},$$

which is true if and only if

$$1 < (\alpha^2 + 1)^{1/2}(S_{k+1}\cdot f(x)),$$

which is true if and only if

$$(S_{k+1}\cdot f(x)) > (\alpha^2 + 1)^{-1/2},$$

which is true if and only if

$$2 - (S_{k+1}\cdot f(x)) < 1 - (\alpha^2 + 1)^{-1/2} = \frac{[\beta(\alpha)]^2}{2},$$

which is true if and only if

$$|f(x) - S_{k+1}|^2 < [\beta(\alpha)]^2.$$

1.9. LEMMA. *If* $f(x)=S_{k+1}$, *then* $I^kS'(x)=0$.

Proof. If $f(x)=S_{k+1}$ then for all $\alpha>0$, $f(x)$ is an element of $C^0(S_{k+1}, \beta(\alpha))$, which by 1.5 and 1.8 means that x is an element of $U(S, \alpha)$. Thus for all $\alpha>0$

$$|I^kS'(x)| < \alpha[S'(x)]_{k+1}.$$

Hence

$$|I^kS'(x)| = 0.$$

1.10. By a Carathéodory Outer Measure (abbreviated C.O.M.) on a metric space X, with distance ρ, we shall mean a non-negative set function Λ defined for all subsets of X, such that

(1) $\Lambda(\varnothing)=0$ ($\varnothing$ is the empty set),

(2) $E_1\subset E_2\subset X$ implies $\Lambda(E_1)\leqq\Lambda(E_2)$,

(3) $E = \bigcup_n E_n$ implies $\Lambda(E) \leqq \sum_n \Lambda(E_n)$,

(4) $\rho(E_1, E_2) > 0$ implies $\Lambda(E_1) + \Lambda(E_2) = \Lambda(E_1 \cup E_2)$.

An outer measure is a non-negative set function defined on the subsets of X which satisfies only the first three above conditions. By a Λ measurable subset of X we shall mean a set E such that for all subsets Q of X,

$$\Lambda(Q) = \Lambda(Q \cap E) + \Lambda(Q \cap CE),$$

where $C(E)$ denotes the complement of E. By a Borel regular C.O.M. we shall mean a C.O.M. such that for any subset E of X there is a Borel set B containing E for which $\Lambda(E) = \Lambda(B)$.

Let A be a closed set in R^n, j a positive integer, Λ an outer measure in R^n. For Y a subset of C^{k+1}, let

$$\Psi(Y) = \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}(Y)]}{r^k},$$

where $\bar{O}$ is the origin of R^n, and $K(x, r)$ is the open sphere of center x and radius r in R^n. *Then* $\psi(Y)$ *is an outer measure in* C^{k+1}.

1.11. LEMMA. *For A closed $\subset R^n$, j is a positive integer, D as in* 1.7, *let*

$$Z = f(A \cap D \cap K(\bar{O}, 1/j)).$$

Then Z is an analytic subset of C^{k+1}, and

$$\Psi(C^{k+1} - Z) = 0.$$

Proof. Since $A \cap D \cap K(\bar{O}, 1/j)$ is a Borel set of R^n, and f is continuous, Z is analytic. Also,

$$C^{k+1} - Z = C^{k+1} \cap Cf[A \cap D \cap K(\bar{O}, 1/j)].$$

Hence $\Psi(C^{k+1} - Z) =$

$$\operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap f^{-1}\{C^{k+1} \cap Cf[A \cap D \cap K(\bar{O}, 1/j)]\}]}{r^k},$$

which is less than or equal to

$$\operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(\bar{O}, r) \cap D \cap C[A \cap D \cap K(\bar{O}, 1/j)]]}{r^k} = 0.$$

1.12. LEMMA. *Let*

$$p(y, \eta) = \frac{\Psi C(y, \eta)}{H^k[C(y, \eta)]},$$

where H^k is Hausdorff k measure. Let

$$H = \left\{ y \mid y \in C^{k+1}, \limsup_{\eta \to 0} p(y, \eta) = +\infty \right\},$$

$$K = \left\{ y \mid y = C^{k+1}, \limsup_{\eta \to 0} p(y, \eta) > 0 \right\}.$$

Then $H^k(K - (H \cup Z)) = 0$.

Proof. By 1.11 $\Psi(C^{k+1} - Z) = 0$. So $\Psi[C(y, \eta) - Z] = 0$. Hence

$$\Psi[C(y, \eta) \cap Z] \leqq \Psi[C(y, \eta)] \leqq \Psi[C(y, \eta) \cap Z] + \Psi[C(y, \eta) - Z].$$

Therefore

$$p(y, \eta) = \frac{\Psi[C(y, \eta) \cap Z]}{H^k[C(y, \eta)]}.$$

Now, by [3; 4],

$$\limsup_{\eta \to 0} \frac{\Psi[C(y, \eta) \cap Z]}{H^k[C(y, \eta)]} = 0 \quad \text{or} \quad +\infty,$$

H^k almost everywhere on $C^{k+1} - Z$. But on $K - H$

$$0 < \limsup_{\eta \to 0} p(y, \eta) < +\infty.$$

So $H^k[K - (H \cup Z)] = 0$.

1.13. The following are immediate consequences of 1.6.

$$\lim_{\eta \to 0} \frac{H^k\left[C\left(y, \frac{\beta(\eta)}{2}\right)\right]}{\alpha(k)\eta^k} = 2^{-k} > 0,$$

$$\lim_{\eta \to 0} \frac{H^k[C(y, \beta(\eta))]}{\alpha(k)\eta^k} = 1,$$

where $\alpha(k)$ is the Lebesgue k measure of the set

$$\{x \mid x \in R^k, |x| \leqq 1\}.$$

1.14. Lemma. *Let*

$H^*(\Lambda, A)$

$$= \left\{ S \mid S \in G_{k+1}, \limsup_{\eta \to 0} \operatorname*{l.u.b.}_{0 < r \leqq 1/j} \frac{\Lambda[A \cap K(\overline{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k) r^k \eta k} = +\infty \right\}.$$

Then S_{k+1} *in* H *implies* S *is in* $H^*(\Lambda, A)$.

Proof. S_{k+1} in H implies

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}C(S_{k+1}, \eta)]}{r^k H^k[C(S_{k+1}, \eta)]} = +\infty.$$

Hence by 1.6

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}C(S_{k+1}, \beta(\eta))]}{r^k H^k[C(S_{k+1}, \beta(\eta))]} = +\infty.$$

Hence since

$f^{-1}[C(S_{k+1}, \beta(\eta))] \subset f^{-1}[C^0(S_{k+1}, 2\beta(\eta))]$,

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}[C^0(S_{k+1}, 2\beta(\eta))]]}{r^k H^k[C(S_{k+1}, \beta(\eta))]} = +\infty,$$

which implies that

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\eta))]]}{r^k H^k\left[C\left(S_{k+1}, \frac{\beta(\eta)}{2}\right)\right]} = +\infty.$$

Hence by 1.14,

$$+\infty = \limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}[C^0(S_{k+1}, \beta(\eta))]]}{\alpha(k) r^k \eta^k},$$

which by 1.5 and 1.8 equals

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k) r^k \eta^k}.$$

Hence S is in $H^*(\Lambda, A)$.

1.15. LEMMA. *Let*

$$K^*(\Lambda, A) = \left\{ S \,\middle|\, S \in G_{k+1}, \limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k) r^k \eta^k} > 0 \right\}.$$

Then S in $K^*(\Lambda, A)$ implies S_{k+1} is in K.

Proof. S in $K^*(\Lambda, A)$ implies

$$0 < \limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k) r^k \eta^k},$$

which by 1.5 and 1.8 is less than or equal to

$$\limsup_{\eta\to 0} \underset{0<r\leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}[C(S_{k+1}, \beta(\eta))]]}{\alpha(k) r^k \eta^k},$$

which by 1.13 is less than or equal to

$$\limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \left\{\frac{\Lambda[A \cap K(\overline{O}, r) \cap f^{-1}[C(S_{k+1}, \beta(\eta))]]}{\alpha(k)r^k\eta^k}\right\} \cdot \left\{\frac{\alpha(k)\eta^k}{H^k[C(S_{k+1}, \beta(\eta))]}\right\},$$

which by 1.6 equals $\limsup_{\eta\to 0} p(S_{k+1}, \eta)$. Hence S_{k+1} is in K.

1.16. Lemma. *Let*

$$E(S) = \{x \mid x \in R^n, I^k S'(x) = \overline{O}\}.$$

Let

$$L^*(A) = \{S \mid S \in G_{k+1}, (A - \overline{O}) \cap K(\overline{O}, 1/j) \cap E(S) \neq \emptyset\}.$$

Then S_{k+1} in Z (see 1.11) *implies that S is in $L^*(A)$.*

Proof. S_{k+1} in Z implies that there is an x in $A \cap D \cap K(\overline{O}, 1/j)$, such that $f(x) = S_{k+1}$. Hence by 1.9, $I^k S'(x) = \overline{O}$, which implies that x is in $E(S)$. Now if x is in $A \cap D$, then x is not the origin. So x is in

$$(A - \overline{O}) \cap K(\overline{O}, 1/j) \cap E(S)$$

which is therefore not empty. Hence S is in $L^*(A)$.

1.17. Theorem.

$$\sigma_{k+1}[K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))] = 0.$$

Proof. By 1.15, if S is in $K^*(\Lambda, A)$, then S_{k+1} is in K. By 1.14, if S is not in $H^*(\Lambda, A)$, then S_{k+1} is not in H. By 1.16, if S is not in $L^*(A)$, then S_{k+1} is not in Z. Hence

$$K^*(\Lambda, A) - (H^*(\Lambda, A) \cup L^*(A))$$

is contained in the set of S in G_{k+1} such that S_{k+1} is in $K - (H \cup Z)$. Hence the theorem follows from the fact that $H^k[K - (H \cup Z)] = 0$ (see 1.12).

II. **Densities.**

2.1. For a fixed $\eta > 0$, a an element of R^n, A a subset of R^n, j a fixed positive integer, g an element of G_n, and Λ a C.O.M., let

$$P(g, \eta, a) = \{x \mid x \in R^n, \mid I^k g(x - a) \mid < \eta \mid I_{n-k} g(x - a) \mid\}.$$

Let

$$M(\Lambda, A, g, \eta, r, a) = \Lambda[A \cap K(a, r) \cap P(g, \eta, a)]/\alpha(k)\eta^k r^k;$$

$$M_j(\Lambda, A, g, \eta, a) = \operatorname*{l.u.b.}_{0<r\leqq 1/j} M(\Lambda, A, g, \eta, r, a);$$

$$U(g, a) = \{x \mid x \in R^n, I^k g(x - a) = \overline{O}\};$$

$$W_j(\Lambda, A) = \left\{(x, g) \mid (x, g) \in A \times G_n, \limsup_{\eta \to 0} M_j(\Lambda, A, g, \eta, x) = +\infty\right\};$$

$$X_j(\Lambda, A) = \left\{(x, g) \mid (x, g) \in A \times G_n, \limsup_{\eta \to 0} M_j(\Lambda, A, g, \eta, x) = 0\right\};$$

$$V_j(A) = \{(x, g) \mid (x, g) \in A \times G_n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\}.$$

2.2. LEMMA. *If A is closed in R^n, then $V_j(A)$ is a Borel set in the cartesian product space $A \times G_n$.*

Proof. For p and q integers, p less than j, and $C(x, r)$ the closed sphere for radius r and center x in R^n, the set $S_{j,p\cdot q}$ of elements of $A \times G_n$ for which

$$A \cap [C(x, 1/j - 1/p) - K(x, (p - j)/(jp + q)] \cap U(g, x) \neq \emptyset$$

is a closed set in $A \times G_n$, and

$$V_j = \bigcup_{p=j+1}^{\infty} \bigcup_{q=1}^{\infty} S_{j,p\cdot q}.$$

2.3. LEMMA. *For A a Borel set of R^n, Λ a C.O.M., $W_j(\Lambda, A)$ and $X_j(\Lambda, A)$ are Borel sets in $A \times G_n$.*

Proof. It suffices to show that

$$\Lambda[A \cap K(x, r) \cap P(g, \eta, x)]$$

is a lower semi-continuous function of (g, η, r, x). For

$$(g_0, \eta_0, r_0, x_0),$$

let λ be any number less than

$$\Lambda[A \cap K(x_0, r_0) \cap P(g_0, \eta_0, x_0)].$$

There exists F, a closed set of R^n, such that

$$F \subset K(x_0, r_0) \cap P(g_0, \eta_0, x_0),$$

and

$$\Lambda(A \cap F) > \lambda,$$

and for (g, η, r, x) sufficiently close to (g_0, η_0, r_0, x_0),

$$F \subset K(x, r) \cap P(g, \eta, x),$$

and

$$\Lambda[A \cap K(x, r) \cap P(g, \eta, x)] \geqq \Lambda(A \cap F) > \lambda.$$

2.4. For A a closed subset of R^n, and x in A, the sets

$$W_j^*(\Lambda, A, x) = \left\{g \mid g \in G_n, \limsup_{\eta\to 0} M_j(\Lambda, A, g, \eta, x) = +\infty\right\},$$

$$X_j^*(\Lambda, A, x) = \left\{g \mid g \in G_n, \limsup_{\eta\to 0} M_j(\Lambda, A, g, \eta, x) = 0\right\},$$

$$V_j^*(A, x) = \{g \mid g \in G_n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\},$$

are Borel sets in G_n.

2.5. For A a closed subset of R^n, g in G_n, the sets

$$\overline{W}_j(\Lambda, A, g) = \left\{x \mid x \in R^n, \limsup_{\eta\to 0} M_j(\Lambda, A, g, \eta, x) = +\infty\right\},$$

$$\overline{X}_j(\Lambda, A, g) = \left\{x \mid x \in R^n, \limsup_{\eta\to 0} M_j(\Lambda, A, g, \eta, x) = 0\right\},$$

$$\overline{V}_j(A, g) = \{x \mid x \in R^n, (A - x) \cap K(x, 1/j) \cap U(g, x) \neq \emptyset\},$$

$$\overline{W}(\Lambda, A, g) = \bigcap_j \overline{W}_j(\Lambda, A, g), \overline{X}(\Lambda, A, g) = \bigcup_j \overline{X}_j(\Lambda, A, g), \text{ and}$$

$$\overline{V}(A, g) = \bigcap_j \overline{V}_j(A, g)$$

are Borel sets in R^n. Also note that $\overline{V}(A, g)$ is the set of all points x of R^n such that x is an accumulation point of $A \cap U(g, x)$.

2.6. For g in G_n, $\eta > 0$, and a in R^n, let

$$Q_i^+(g, \eta) = \{x \mid x \in R^n, \mid I^k g(x - a) \mid < (n - k)^{1/2}\eta[g(x - a)]_i\} \quad \text{for } k + 1 \leqq i \leqq n.$$

Likewise let

$$Q_i^-(g, \eta) = \{x \mid x \in R^n, \mid I^k g(x - a) \mid < -(n - k)^{1/2}\eta[g(x - a)]_i\} \quad \text{for } k + 1 \leqq i \leqq n.$$

Now let

$$F_i^+ = \left\{g \mid g \in G_n, \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^+(g, \eta)]}{\alpha(k) r^k \eta^k} > 0\right\},$$

and let

$$F_i^- = \left\{g \mid g \in G_n, \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^-(g, \eta)]}{\alpha(k) r^k \eta^k} > 0\right\}.$$

F_i^+ and F_i^- are Borel sets in G. The proof of this is essentially the same as that given in 2.3.

2.7. LEMMA.

$$P(g, \eta, a) \subset \bigcup_{i=k+1}^{n} [Q_i^+(g, \eta) \cup Q_i^-(g, \eta)].$$

Proof. If not, then there exists x in R^n such that

$$| I^k g(x - a) | < \eta | I_{n-k}\, g(x - a) |$$

and for all i, $k+1 \leqq i \leqq n$,

$$| I^k g(x - a) | \geqq (n - k)^{1/2}\eta | [g(x - a)]_i | .$$

Hence

$$(n - k) | I^k g(x - a) |^2 \geqq (n - k)\eta^2 \left[\sum_{i=k+1}^{n} [g(x - a)]_i^2 \right]$$
$$= (n - k)\eta^2 | I_{n-k} g(x - a) |^2.$$

This is a contradiction.

2.8. LEMMA. *Let $I(p, q)$ be the identity matrix with the pth row replaced by the qth row, and the qth row replaced by the negative of the pth row. Then, for $k+1 \leqq p, q \leqq n$,*

$$I(p, q) \cdot [F_p^+ - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = F_q^- - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a)),$$

where "$\cdot$" denotes cosetting with respect to $I(p, q)$.

Proof. For g in G_n, and $i \neq p, q$,

$$[I(pq)g]_i = g_i; \qquad [I(p, q)g]_p = g_q; \qquad [I(p, q)g]_q = - g_p.$$

Thus, since $k+1 \leqq p, q \leqq n$, we have

$$| I^k(I(p, q)g)(x - a) | = | I^k g(x - a) | ,$$

and

$$| I_{n-k}(I(p, q)g)(x - a) | = | I_{n-k} g(x - a) | .$$

Thus $Q_p^+(g, \eta) = Q_p^-(I(p, q)g, \eta)$, and hence

$$F_q^- = I(p, q) \cdot F_p^+.$$

Also $P(g, \eta, a) = P(I(p, q)g, \eta, a)$, and hence

$$W_j^*(\Lambda, A, a) = I(p, q) \cdot W_j^*(\Lambda, A, a).$$

Also $U(g, a) = U(I(p, q)g, a)$, and hence

$$V_j^*(A, a) = I(p, q)\cdot V_j^*(A, a).$$

Hence

$$I(p, q)\cdot[\overset{+}{F_p} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = \overset{-}{F_j} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a).$$

2.9. LEMMA. *For* $k+1 \leqq p, q \leqq n$,

$$\sigma_n[\overset{+}{F_p} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))] = \sigma_n[\overset{-}{F_q} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))].$$

Proof. By 2.8

$$\overset{-}{F_q} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a)) = I(p, q)\cdot[\overset{+}{F_p} - (V_j^*(A, a) \cup W_j^*(\Lambda, A, a))],$$

and the lemma follows since σ_n is the Haar measure in G_n.

III. **Further density considerations.**

3.1. Let A be a closed subset of R^n, g an element of G_n, and a an element of A. Let

$$t(x) = g(x - a), \qquad \text{for } x \text{ an element of } R^n.$$

Then t is a distance preserving homeomorphism of R^n onto R^n such that $t(a) = \overline{O}$, $tK(a, r) = K(\overline{O}, r)$, and $t(A)$ is a closed subset of R^n.

3.2. LEMMA. *For* s *in* G_{k+1}, $\eta > 0$,

$$\overset{+}{Q}_{k+1}(S'g, \eta) = t^{-1}[\mathfrak{U}(S, (n-k)^{1/2}\eta)].$$

Proof. $x \in \overset{+}{Q}_{k+1}(S'g, \eta)$ if and only if

$$| I^k S'g(x-a) | < (n-k)^{1/2}\eta[S'g(x-a)]_{k+1},$$

which is true if and only if

$$| I^k S'g(x-a) | < (n-k)^{1/2}\eta[S'(t(x))]_{k+1},$$

which is true if and only if $t(x)$ is in $\mathfrak{U}(S, (n-k)^{1/2}\eta)$.

3.3. LEMMA. *For* g *in* G_n, a *in* R^n, S *in* G_{k+1}, $\eta > 0$,

$$t^{-1}(\mathfrak{U}(S, \eta)) \subset P(S'g, \eta, a).$$

Proof. If x is in $t^{-1}(\mathfrak{U}(S, \eta))$, then $t(x)$ is in $\mathfrak{U}(S, \eta)$, and therefore

$$| I^k S'g(x-a) | < \eta[S'g(x-a)]_{k+1},$$

and hence

$$|I^k S'g(x - a)|^2 < \eta^2 [S'g(x-a)]^2_{k+1} \leqq \eta^2 |I_{n-k} S'g(x-a)|^2,$$

and therefore x is in $P(S'g, \eta, a)$.

3.4. LEMMA. *For g in G_n, a in R^n, S in G_{k+1},*

$$t^{-1}(E(S)) = \mathfrak{U}(S'g, a).$$

Proof. x is in $t^{-1}(E(S))$ if and only if $t(x)$ is in $E(S)$, which is true if and only if

$$|I^k S'g(x - a)| = 0,$$

which is true if and only if x is in $\mathfrak{U}(S'g, a)$.

3.5. LEMMA. *For A a closed subset of R^n, a an element of A, g in G_n, S in G_{k+1}; S is in $L^*(t(A))$ if and only if $S'g$ is in $V_j^*(A, a)$.*

Proof. By 3.4

$$(A - a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a) = t^{-1}[(t(A) - \overline{O}) \cap K(0, 1/j) \cap E(S)].$$

So S is in $L^*(t(A))$ if and only if

$$(t(A) - \overline{O}) \cap K(\overline{O}, 1/j) \cap E(S) \neq \varnothing,$$

which is true if and only if

$$t[(A - a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a)] \neq \varnothing,$$

which is true if and only if

$$(A - a) \cap K(a, 1/j) \cap \mathfrak{U}(S'g, a) \neq \varnothing,$$

which is true if and only if $S'g$ is in $V_j^*(A, a)$.

3.6. For Λ an outer measure, and E a subset of R^n, let

$$\Lambda^*(E) = \Lambda[t^{-1}(E)].$$

Then Λ^* is an outer measure in R^n.

3.7. LEMMA. *For Λ an outer measure, A a closed subset of R^n, g in G_n, S in G_{k+1}; if S is in $H^*(\Lambda^*, t(A))$, then $S'g$ is in $W_j^*(\Lambda, A, a)$.*

Proof. If S is in $H^*(\Lambda^*, t(A))$, then

$$+\infty = \limsup_{\eta \to 0} \ \underset{0 < r \leqq 1/j}{\text{l.u.b.}} \frac{\Lambda^*[t(A) \cap K(\overline{O}, r) \cap \mathfrak{U}(S, \eta)]}{\alpha(k) r^k \eta^k},$$

which by 3.3 is less than or equal to

$$\limsup_{\eta \to 0} \ \underset{0 < r \leqq 1/j}{\text{l.u.b.}} \frac{\Lambda[A \cap K(a, r) \cap P(S'g, \eta, a)]}{\alpha(k) r^k \eta^k}.$$

Hence $S'g$ is in $W_j^*(\Lambda, A, a)$.

3.8. LEMMA. *If $S'g$ is in $\overset{+}{F}_{k+1}$, then S is in $K^*(\Lambda^*, t(A))$.*

Proof. If $S'g$ is in $\overset{+}{F}_{k+1}$, then

$$0 < \limsup_{\eta \to 0} \ \underset{0<r\leq 1/j}{\text{l.u.b.}} \ \frac{\Lambda[A \cap K(a, r) \cap \overset{+}{Q}_{k+1}(S'g, \eta)]}{\alpha(k) r^k \eta^k}$$

which by 3.2 equals

$$\limsup_{\eta \to 0} \ \underset{0<r\leq 1/j}{\text{l.u.b.}} \ \frac{\Lambda^*[t(A) \cap K(\overline{O}, r) \cap \mathfrak{U}(S, (n-k)^{1/2}\eta)]}{\alpha(k) r^k \eta^k},$$

and hence S is in $K^*(\Lambda^*, t(A))$.

3.9. THEOREM. *For any g in G_n, and a in A, a closed subset of R^n, and Λ an outer measure,*

$$\sigma_{k+1}\{S \mid S'g \in [\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} = 0.$$

Proof. By 3.5, 3.7, and 3.8,

$$\{S \mid S'g \in [\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\}$$
$$\subset K^*[\Lambda^*, t(A)] - \{H^*[\Lambda^*, t(A)] \cup L^*[t(A)]\}.$$

Since Λ^* is an outer measure, and $t(A)$ is closed in R^n, we may apply 1.17.

3.10. THEOREM. *For $a \in A$, a closed subset of R^n, and Λ a C.O.M.*

$$\sigma_n[\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

Proof. Let α: $(G_{k+1} \times G_n) \to G_n$ be a mapping defined by $\alpha(S, g) = S'g$. α is a continuous mapping. Hence the set

$$\{(S, g) \mid (S, g) \in G_{k+1} \times G_n, S'g \in [\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\}$$
$$= \alpha^{-1}[\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]$$

is a Borel set in $G_{k+1} \times G_n$.

Let $c(S, g)$ equal 1 if $S'g$ is in $\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))$, and 0 otherwise. Then, since σ_n is a Haar measure in G_n, for any S in G_{k+1},

$$\sigma_n[\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]$$
$$= \sigma_n\{g \mid S'g \in [\overset{+}{F}_{k+1} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]\} = \int_{G_n} c(S, g) d\sigma_n.$$

So by 3.9

$$\sigma_n[F_{k+1}^{+} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]$$
$$= \int_{G_{k+1}} \int_{G_n} c(S, g) d\sigma_n d\sigma_{k+1} = \int_{G_n} \int_{G_{k+1}} c(S, g) d\sigma_{k+1} d\sigma_n = 0.$$

3.11. LEMMA. *For* $k+1 \leqq p,\ q \leqq n$, a, A, *and* Λ *as in* 3.10,

$$\sigma_n[F_p^{+} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = \sigma_n[F_q^{-} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

Proof. By 2.8 and 3.10,

$$\sigma_n[F_p^{+} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = \sigma_n[F_q^{-} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))]$$
$$= \sigma_n[F_{k+1}^{+} - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] = 0.$$

3.12. LEMMA. *For* a, A, *and* Λ *as in* 3.10

$$G_n - X_j^*(\Lambda, A, a) \subset \bigcup_{i=k+1}^{n} (F_i^{+} \cup F_i^{-}).$$

Proof. If not, then there exists a g in G_n, such that

$$\limsup_{\eta\to 0} M_j(\Lambda, A, g, \eta, a) > 0,$$

and for all i, $k+1 \leqq i \leqq n$,

$$0 = \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^{+}(g, \eta)]}{\alpha(k) r^k \eta^k}$$
$$= \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^{-}(g, \eta)]}{\alpha(k) r^k \eta^k}.$$

Hence by 2.7,

$$0 < \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap P(g, \eta, a)]}{\alpha(k) r^k \eta^k}$$
$$\leqq \sum_{i=k+1}^{n} \left[\limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^{+}(g, \eta)]}{\alpha(k) r^k \eta^k} + \limsup_{\eta\to 0} \operatorname*{l.u.b.}_{0<r\leqq 1/j} \frac{\Lambda[A \cap K(a, r) \cap Q_i^{-}(g, \eta)]}{\alpha(k) r^k \eta^k} \right] = 0.$$

This is a contradiction.

3.13. LEMMA. *For* a, A *and* Λ *as in* 3.10

$$\sigma_n[G_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(A, a)] = 0.$$

Proof. By 3.12

$$G_n - (X_j^*(\Lambda, A, a) \cup W_j^*(\Lambda, A, a) \cup V_j^*(A, a))$$

$$\subset \bigcup_{i=k+1}^{n} [F_i^+ - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))] \cup [F_i^- - (W_j^*(\Lambda, A, a) \cup V_j^*(A, a))],$$

and the result follows from 3.11.

3.14. Lemma. *For A, a closed subset of R^n, j an integer, Λ a C.O.M.,*

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0$$

for σ_n almost every g in G_n.

Proof. By 3.13 for any a in A,

$$\sigma_n[G_n - (W_j^*(\Lambda, A, a) \cup X_j^*(\Lambda, A, g) \cup V_j^*(A, a))] = 0.$$

Apply Fubini's theorem to the characteristic function of

$$(A \times G_n) - (W_j(\Lambda, A) \cup X_j(\Lambda, A) \cup V_j(A)).$$

3.15. Theorem. *For A a closed subset of R^n, Λ a C.O.M.*

$$\Lambda[A - (\overline{W}(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n.

Proof. By 3.14, for every integer j, there is a subset Z_j of G_n, such that $\sigma_n(Z_j) = 0$, and for g in $G_n - Z_j$

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0.$$

Let $Z = \bigcup_{j=1}^{\infty} Z_j$. Then $\sigma_n(Z) = 0$, and for g in $G_n - Z$, and for all integers j,

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}_j(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0,$$

and thus, by 2.5,

$$\Lambda[A - (\overline{W}_j(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}_j(A, g))] = 0.$$

Also, $\overline{W}_j(\Lambda, A, g)$ and $\overline{V}_j(A, g)$ are monotone decreasing sequences of sets. So

$$A - (\overline{W}(\Lambda, A, g) \cup \overline{X}(\Lambda, A, g) \cup \overline{V}(A, g))$$

$$= [A - \overline{X}(\Lambda\ A, g)] \cap \left[\bigcup_j C\overline{W}_j(\Lambda, A, g) \right] \cap \left[\bigcup_j C\overline{V}_j(A, g) \right]$$

$$\subset [A - \overline{X}(\Lambda, A, g)] \cap \bigcup_j [C\overline{W}_j(\Lambda, A, g) \cap C\overline{V}_j(A, g)]$$

$$= \bigcup_j [A - \overline{X}(\Lambda, A, g) \cup \overline{W}_j(\Lambda, A, g) \cup \overline{V}_j(A, g)],$$

and the result follows.

IV. (Λ, k) Unrectifiable sets.

4.1. By a Lipschitz mapping T from a metric space X, with metric ρ, to a metric space Y, with metric δ, we shall mean a mapping such that there exists a constant M such that for all x, y in X, $\delta[T(x), T(y)] \leqq M\rho(x, y)$.

A subset of Euclidean n-space will be said to be k-rectifiable if it is the image under a Lipschitz mapping of a bounded subset of Euclidean k-space.

If Λ is an outer measure on Eucidean n-space, a subset E of n-space will be said to be (Λ, k) unrectifiable if every k-rectifiable subset of E is of Λ measure 0.

LEMMA. *Let Λ be an outer measure in R^n. Let A be a subset of R^n such that $d(A)$, the diameter of A is less than $1/j$, for a fixed integer j. Let g be an element of G_n, $0<\delta<1$, $0<\gamma<+\infty$, and assume that*

$$(1) \qquad M_j(\Lambda, A, g, \eta, x) < \gamma \qquad \text{for } x \text{ in } A \text{ and } 0 < \eta < \delta.$$

Let

$$(2) \quad B = \{x \mid x \in A, A \cap P(g, \eta, x) \cap K(x, r) \neq \emptyset \text{ for } r > 0, 0 < \eta < 1\}.$$

For x in R^n, let

$$x' = I_{n-k}g(x);$$

and let

$$x'' = I^k g(x).$$

Also let

$$K''(x'', r) = I^k g(K(x, r)).$$

Then for a in B, we have

$$(3) \qquad \Lambda[B \cap (I^k g)^{-1}(K''(a'', r))] < 2^{k+1}10^{2k}\gamma\alpha(k)r^k,$$

for $0<r<\delta/12j$.

Proof. Since $d(A)<1/j$, (1) implies that

$$(4) \qquad M(\Lambda, A, g, \eta, r, x) < \gamma \qquad \text{for } x \text{ in } A, 0 < \eta < \delta, r > 0.$$

Now for x, y in R^n

$$|y'' - x''| = |I^k g(y) - I^k g(x)| = |I^k g(y - x)|,$$
$$|y' - x'| = |I_{n-k}g(y) - I_{n-k}g(x)| = |I_{n-k}g(y - x)|.$$

With this remark, the proof proceeds exactly as in ([1], 5.2). For a in B, $0<r<\delta/12j$, let $\eta=12rj$, $\epsilon=\eta/12=rj$. Set

$$E = B \cap (I^k g)^{-1}(K''(a'', r)).$$

For x in E, set $h(x) = \text{l.u.b.}\ |y'-x'|$ for y in

$$A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K''(a'', r)).$$

Then $0 < h(x) \leqq 1/j$. For x in E, let $\bar{x}$ in $A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K''(a'', r))$ be such that

$$12|\bar{x}' - x'| > 11h(x). \tag{5}$$

Since $\bar{x}$ is in $P(g, \epsilon, x)$,

$$|\bar{x}'' - x''| < \epsilon|\bar{x}' - x'|. \tag{6}$$

For x in E, set

$$\begin{gathered} Q(x) \text{ equal the set of } y \text{ in } E \text{ such that} \\ |y'' - x''| < 5\epsilon h(x). \end{gathered} \tag{7}$$

The remainder of the proof is divided into five parts.

PART 1. For x in E, $Q(x) \subset P(g, \eta, x) \cup P(g, \eta, \bar{x})$.

Proof. If not, then there is an x in E, such that

$$|y'' - x''| < 5\epsilon h(x),\ |y'' - x''| \geqq \eta|y' - x'|,\ |y'' - \bar{x}''| \geqq \eta|y' - \bar{x}'|.$$

Then

$$\begin{aligned} 0 < 11\epsilon h(x) &< 12\epsilon|\bar{x}' - x'| = \eta|\bar{x}' - x'| \leqq \eta|\bar{x}' - y'| + \eta|y' - x'| \\ &\leqq |y'' - \bar{x}''| + |y'' - x''| \leqq |x'' - \bar{x}''| + 2|y'' - x''| < \epsilon|x' - \bar{x}'| \\ &\quad + 10\epsilon h(x) \leqq 11\epsilon h(x). \end{aligned}$$

This is a contradiction.

PART 2. For x in E, $K''(x'', \epsilon h(x)) \subset K''(a'', 2r)$.

Proof. If $|y''-x''| < \epsilon h(x)$, then, since $|x''-a''| < r$,

$$|y'' - a''| < |y'' - x''| + |x'' - a''| < \epsilon h(x) + r < 2r.$$

PART 3. If y is in $Q(x)$, then $|y'-x'| < 5h(x)$.

Proof. Assume that there is a y in $Q(x)$ such that $|y'-x'| \geqq 5h(x)$. Then $|y''-x''| < 5\epsilon h(x) \leqq \epsilon|y'-x'|$. Thus y is in $A \cap P(g, \epsilon, x) \cap (I^k g)^{-1}(K(a'', r))$, and hence $1/j \geqq h(x) \geqq 5h(x) > 0$. This is a contradiction.

PART 4. If x is in E, then $\Lambda(Q(x)) \leqq 2(84)^k \gamma\alpha(k)[\epsilon h(x)]^k$.

Proof. By Part 3, if y is in $Q(x)$, then

$$|y - x| \leqq |y' - x'| + |y'' - x''| < 5h(x) + 5\epsilon h(x) = 5(\epsilon + 1)h(x) < 6h(x).$$

Also

$$\begin{aligned} |y - \bar{x}| &\leqq |y - x| + |x - \bar{x}| \leqq 5(\epsilon + 1)h(x) + |x' - \bar{x}'| + |x'' - \bar{x}''| \\ &< 5(\epsilon + 1)h(x) + (\epsilon + 1)|x' - \bar{x}| < 6(\epsilon + 1)h(x) \\ &< (13/12)\cdot 6h(x) < 7h(x). \end{aligned}$$

Hence

$$Q(x) \subset K(x, 7h(x)) \cap K(\bar{x}, 7h(x)).$$

Hence, by Part 1,

$$Q(x) \subset [A \cap P(g, \eta, x) \cap K(x, 7h(x))] \cup [A \cap P(g, \eta, \bar{x}) \cap K(\bar{x}, 7h(x))].$$

Thus by (4) and since

$$\eta = 12\epsilon, \quad \Lambda[Q(x)] \leqq 2\gamma\alpha(k)(7h(x)\eta)^k = 2(84)^k\gamma\alpha(k)(\epsilon h(x))^k.$$

PART 5. $\Lambda(E) \leqq 2^{k+1}10^{2k}\alpha(k)\gamma r^k$.

Proof. Since, for x in E, $I^k g(E) \subset \bigcup K''(x'', \epsilon h(x))$, by a covering theorem of Morse, [5], there are $x_1, x_2, \cdots$ in E such that $I^k g(E) \subset \bigcup_{i=1}^\infty K''(x_i'', 5\epsilon h(x_k))$, and for $p \neq q$

$$K''(x_p'', \epsilon h(x_p)) \cap K''(x_q'', \epsilon h(x_q)) = \emptyset.$$

Now for x in E, there is an x_i such that $|x'' - x_i''| < 5\epsilon h(x)$, and hence x is in $Q(x_i)$. Thus

$$E \subset \bigcup_{i=1}^\infty Q(x_i).$$

From Parts 2 and 4,

$$\Lambda(E) \leqq \sum_{i=1}^\infty \Lambda(Q(x_i)) \leqq 2(84)^k\alpha(k)\gamma \sum_{i=1}^\infty (\epsilon h(x_i))^k$$
$$\leqq 2(84)^k(2)^k\gamma\alpha(k)r^k \leqq 2^{k+1}10^{2k}\gamma\alpha(k)r^k.$$

4.2. THEOREM. *Let Λ be a C.O.M. in R^n, g in G_n, and let A be a (Λ, k) unrectifiable subset of R. Then (see 2.5)*

$$\Lambda[\overline{X}(\Lambda, A, g)] = 0.$$

Proof. It suffices to show that for j an integer $\Lambda[\overline{X}_j(\Lambda, A, g)] = 0$. Also, since R^n is separable, we may assume $d(\overline{X}_j(\Lambda, A, g)) < 1/j$. Now $\limsup_{\eta\to 0} M_j(\Lambda, \overline{X}_j, g, \eta, x) = 0$ for x in $\overline{X}_j$. Assume $\Lambda[\overline{X}_j(\Lambda, A, g)] > 0$. Hence by 2.5, there is a set $\overline{A}_j$ contained in $\overline{X}_j$, and a sequence of positive n numbers $\delta_1, \delta_2, \cdots$, such that

$$\Lambda(\overline{A}_j) > 0, \ \overline{M}_j(\Lambda, \overline{A}_j, g, \eta, x) < 1/i, \qquad \text{for } 0 < \eta < \delta_i$$

and x in $\overline{A}_j$. Let B_j be the set of x in $\overline{A}_j$, such that for $r > 0$, $0 < \eta < 1$,

$$\overline{A}_j \cap P(g, \eta, x) \cap K(x, r) \neq \emptyset.$$

Since $\overline{A}_j$ is (Λ, k) unrectifiable, by Federer ([2] 4.3 Theorem),

$$\Lambda(\overline{A}_j - B_j) = 0, \qquad \Lambda[B_j] = \Lambda[\overline{A}_j] > 0.$$

Thus there is a point a in B_j such that

(1) $$\limsup_{r\to 0} \Lambda[B \cap (I^k g)^{-1}(K''(a'', r))]/\alpha(k)r^k > \lambda > 0,$$

for some λ. Choose i such that $1/i<\lambda/2^{k+1}10^{2k}$. By 4.1,

(2) $$\Lambda[B \cap (I^k g)^{-1}(K''(a'', r))] < \alpha(k)r^k\lambda,$$

for $0<r<\delta_i/12j$.

Since (1) contradicts (2), it follows that

$$\Lambda(\overline{X}_j(\Lambda, A, g)) = 0.$$

V. Lebesgue measure of projected sets.

5.1. Lemma. *If Λ is a C.O.M. in R^n, g in G_n, and A is a subset of R^n, with $\Lambda(A)<+\infty$, then,*

$$L_k[I^k g(\overline{W}(\Lambda, A, g))] = 0,$$

where L_k denotes Lebesgue measure in R^k.

Proof. For E in R^k, let

$$\Lambda_g(E) = \Lambda[A \cap (I^k g)^{-1}(E)].$$

Then Λ_g is finite-valued C.O.M. in R^k, and for x'' in R^k

(1) $$\limsup_{t\to 0} \Lambda_g(K''(x'', t))/\alpha(k)t^k < +\infty,$$

for L_k almost every x'' in R^k. Now for x in $\overline{W}(\Lambda, A, g)$, and $0<\eta<1$, $0<r<+\infty$, the set $P(g, \eta, x)\cap K(x, r)$ is contained in the set of y in R^n such that y'' is an element of $K''(x'', \eta r)$. Thus there is a sequence

$$(\eta_i, r_i) \to (0, 0) \qquad \text{for } i \to +\infty,$$

such that

$$+\infty = \lim_{i\to\infty} M(\Lambda, A, g, \eta_i, r_i, x) = \lim_{i\to\infty} \Lambda_g(K''(x'', \eta_i r_i))/\alpha(k)\eta_i^k r_i^k,$$

and therefore the result follows from (1).

5.2. Lemma. *Let Λ be a C.O.M. in R^n, g in G_n, and A a Borel subset of R^n such that*

$$\Lambda(A) < +\infty$$

and

$$\Lambda(A) \geqq \int_{R_k} N(x'', I^k g, A)dL_k,$$

where $N(x'', I^k g, A)$ is the number of points (possibly ∞) in $A \cap (I^k g)^{-1}(x'')$. Then

$$L_k(I^k g(\overline{V}(A, g))) = 0.$$

Proof. $N(x'', I^k g, A)$ is finite for L_k almost every x'' in R^k. But x in $\overline{V}(A, g)$ implies $N(x'', I^k g, A)$ is infinite. Hence the result follows.

VI. **The μ_n^k measure.**

6.1. Let $\mathcal{Z}$ denote the family of subsets Z of G_n, such that $\sigma_n(Z) = 0$. For B a Borel subset of R^n, and Z in $\mathcal{Z}$, let

$$\overset{*}{\lambda}_Z(B) = \text{l.u.b. } L_k[I^k g(B)], \qquad \text{for } g \text{ in } G_n - Z.$$

Let

$$\lambda(B) = \text{g.l.b. } \overset{*}{\lambda}_Z(B), \qquad \text{for } Z \text{ in } \mathcal{Z}.$$

For E a subset of R^n, and $\epsilon > 0$, let $\mu_\epsilon(E) = \text{g.l.b. } \sum \lambda(B_i)$, the sum being taken over a countable covering of E by Borel sets B_i such that $d(B_i) < \epsilon$, and the greatest lower bound being taken over the family of all such coverings.

For E a subset of R^n, let

$$\mu_n^k(E) = \lim_{\epsilon \to 0} \mu_\epsilon(E).$$

μ_n^k is a Borel regular C.O.M. in R^n.

6.2. Lemma. *Let B_i be Borel sets for $i = 0, 1, 2, \cdots$, and assume $B_0 \subset \bigcup_i B_i$, and Z is in $\mathcal{Z}$. Then*

$$\lambda_Z^*(B_0) \leq \sum_i \overset{*}{\lambda}_Z(B_i).$$

Proof.

$$\begin{aligned} \text{l.u.b. } L_k[I^k g(B_0)] &\leq \text{l.u.b. } L_k\left[\bigcup_i [I^k g(B_i)]\right] \leq \text{l.u.b. } \sum_i L_k[I^k g(B_i)] \\ &\leq \sum_i \text{l.u.b. } L_k[I^k g(B_i)], \end{aligned}$$

all least upper bounds being taken over g in $G_n - Z$.

6.3. Lemma. *For B a Borel set in R^n, there is a set Z in $\mathcal{Z}$ such that* $\lambda(B) = \lambda_Z^*(B)$.

Proof. For any integer i, there is a set Z_i in $\mathcal{Z}$ such that $\lambda_{Z_i}^*(B) < \lambda(B) + 1/i$. Let $Z = \bigcup_i Z_i$. Z is in $\mathcal{Z}$, and $\lambda(B) \leq \lambda_Z^*(B) < \lambda_{Z_i}^*(B) < \lambda(B) + 1/i$. Hence for all i, $\lambda(B) \leq \lambda_Z^*(B) < \lambda(B) + 1/i$. Therefore $\lambda(B) = \lambda_Z^*(B)$.

6.4. Lemma. *For $E \subset R^n$, there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$, $\mu_n^k(E) \geqq L_k I^k g(E)$.*

Proof. Since μ_n^k is Borel regular, it suffices to work with Borel sets. Let B be a Borel set, $\epsilon > 0$, and i an integer. There is a countable covering of B by Borel sets B_j, $d(B_j) > \epsilon$, such that $\mu_n^k(B) + 1/i > \sum_{i=1}^{\infty} \lambda(B_j)$. By 6.3 for each j there is a Z_j in $\mathcal{Z}$ such that

$$\lambda(B_j) = \lambda^* Z_j(B_j).$$

Set $Z^i = \bigcup_{j=1}^{\infty} Z_j$. Then Z^i is in $\mathcal{Z}$ and, (by 6.2)

$$\mu_n^k(B) + 1/i > \sum_{i=1}^{\infty} \lambda_{Z^i}^*(B_j) \geqq \lambda_{Z^i}^*(B).$$

Now set $Z = \bigcup_{i=1}^{\infty} Z^i$. Then Z is in $\mathcal{Z}$, and

$$\mu_n^k(B) + 1/i > \lambda_Z^*(B) \geqq L_k I^k g(B),$$

for g in $G_n - Z$, and for all i. Hence $\mu_n^k(B) \geqq L_k I^k g(B)$ for g in $G_n - Z$.

6.5. Lemma. *Let Λ be a Borel regular C.O.M. in R^n. Assume that for any subset E of R^n, there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$, $\Lambda(E) \geqq L_k I^k g(E)$. Then $\Lambda(E) \geqq \mu_n^k(E)$.*

Proof. Again it suffices to work only with Borel sets B. For $\epsilon > 0$, B may be written as a countable union of Borel sets B_i, where $d(B_i) < \epsilon$, and for $i \neq j$, $B_i \cap B_j = \emptyset$. Further for all integers i, there is a set Z_i in $\mathcal{Z}$ such that for g in $G_n - Z_i$. $\Lambda(B_i) \geqq L_k I^k g(B_i)$. Set $Z = \bigcup_{i=1}^{\infty} Z_i$. Then Z is in $\mathcal{Z}$, and $\Lambda(B_i) \geqq \text{l.u.b.}\ [L_k I^k g(B_i)] \geqq \lambda(B_i)$ for g in $G_n - Z$. Hence $\Lambda(B) \geqq \sum_{i=1}^{\infty} \lambda(B_i) \geqq \mu_\epsilon(B)$. The result follows since ϵ was arbitrary.

6.6. Lemma. *Let Λ be a Borel regular C.O.M. in R^n. Let Γ be a Borel regular C.O.M. in R^k such that for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z_E in $\mathcal{Z}$ such that for g in $G - Z_E$, $\Lambda(E) \geqq \Gamma I^k g(E)$. Then for all $E \subset R^n$, such that $\Lambda(E) < +\infty$, there is a set Z in $\mathcal{Z}$ such that for all subsets $E^* \subset E$, and g in $G_n - Z$,*

$$\Lambda(E^*) \geqq \Gamma[I^k g(E^*)].$$

Proof. Again it suffices to consider only Borel sets B in R^n, and Borel subsets $B^* \subset B$. First, let θ be the family of sets C such that C is the union of a finite number of spheres of rational center and rational radius in R^n. θ is a countable family. If F is a compact subset of R^n, then there exists $C_1 \supset C_2 \supset \cdots$, $C_i \in \theta$, $F = \bigcap_{i=1}^{\infty} C_i$. We may further assume that B is bounded. Let

$$\theta_B = \{E \mid E = B \cap C,\ C \in \theta\}.$$

θ_B is countable. For each E in θ_B, there is a Z_E in $\mathcal{Z}$ such that, for g in $G_n - Z_E$,

$\Lambda(E) \geqq \Gamma[I^k g(E)]$. Set $Z = \bigcup Z_E$, for E in θ_B. Z is in $\mathcal{Z}$, and for E in θ_B and g in $G_n - Z$,

$$\Lambda(E) \geqq \Gamma[I^k g(E)].$$

Now take F closed with respect to B. Then there is a compact set $F^* \subset R^n$, such that $F = F^* \cap B$. Hence there is a sequence of sets $C_1 \supset C_2 \supset \cdots$ such that $F^* = \bigcap_{i=1}^{\infty} C_i$. Set $E_i = C_i \cap B$. $F = \bigcap_{i=1}^{\infty} E_i$, and $E_1 \supset E_2 \supset \cdots$. Also for all integers i, and for g in $G_n - Z$,

$$\Gamma[I^k g(F)] \leqq \Gamma[I^k g(E_i)] \leqq \Lambda(E_i).$$

Since $\lim_{i \to \infty} \Lambda(E_i) = \Lambda(F)$, we have, for all g in $G_n - Z$,

$$\Gamma[I^k g(F)] \leqq \Lambda(F).$$

Now let O be open with respect to B. Then there is a bounded open set O^* in R^n, such that $O = O^* \cap B$, and a sequence of compact sets $F_1^* \subset F_2^* \subset \cdots$, such that $O^* = \bigcup_{i=1}^{\infty} F_i^*$. Set $F_i = F_i^* \cap B$. F_i is closed relative to B, and $O = \bigcup_{i=1}^{\infty} F_i$. For all integers i, and g in $G_n - Z$,

$$\Gamma[I^k g(F_i)] \leqq \Lambda(F_i) \leqq \Lambda(O).$$

Now $I^k g(F_i) \subset I^k g(F_{i+1})$, and $I^k g(F_i)$ is Γ measurable, and $I^k g(O) = \bigcup_{i=1}^{\infty} I^k g(F_i)$. So we have that $\Gamma[I^k g(F_i)] \to \Gamma[I^k g(O)]$ as $i \to +\infty$, and thus $\Gamma[I^k g(O)] \leqq \Lambda(O)$. Now take B^* a Borel subset of B. Fix $\epsilon > 0$. Then there is a set O open relative to B, such that $O \supset B^*$, and

$$\Lambda(O) < \Lambda(B^*) + \epsilon.$$

Then for g in $G_n - Z$,

$$\Gamma[I^k g(B^*)] \leqq \Gamma[I^k g(O)] \leqq \Lambda(O) < \Lambda(B^*) + \epsilon.$$

Thus

$$\Gamma[I^k g(B^*)] \leqq \Lambda(B^*).$$

6.7. Lemma. *For E a subset of R^n, such that $\mu_n^k(E) < +\infty$, there is a set Z in $\mathcal{Z}$ such that for g in $G_n - Z$,*

$$\mu_n^k(E) \geqq \int_{R^k} N(x'', I^k g, E) dL_k.$$

Proof. Again it suffices to work with Borel sets $B \subset R^n$. By 6.6 there is a set Z in $\mathcal{Z}$ such that for all Borel subsets $B^* \subset B$, and g in $G_n - Z$, $\mu_n^k(B^*) \geqq L_k I^k g(B^*)$. For i an integer, R^n may be written as the union of a countable family of disjoint Borel sets B_{ij}, where $d(B_{ij}) < 1/i$. For x'' in R^k, let $f_{ij}(x'')$ be the characteristic function of $I^k g(B \cap B_{ij})$, and let $f_i(x'') = \sum_{i=1}^{\infty} f_{ij}(x'')$. Then

$$\int_{R^k} f_i(x'')dL_k = \sum_{j=1}^{\infty}\int_{R^k} f_{ij}(x'')dL_k = \sum_{j=1}^{\infty} L_k[I^k g(B \cap B_{ij})]$$
$$\leqq \sum_{j=1}^{\infty} \mu_n^k(B \cap B_{ij}) = \mu_n^k(B).$$

Since $f_i(x'') \to N(x'', I^k g, B)$ as $i \to +\infty$, the result follows from the lemma of Fatou.

6.8. LEMMA. *Let E be a k-rectifiable set in R^n. Then*

$$H^k(E) = \mu_n^k(E).$$

Proof. Again it suffices to work with Borel sets B. Let Q denote the unit cube in R^k, that is, for x in R^n, and x in Q, and $x = (x_1, x_2, \cdots, x_n)$, we have

$$0 \leqq x_i \leqq 1, \qquad \text{for } 1 \leqq i \leqq k,$$

and

$$x_i = 0, \qquad \text{for } k+1 \leqq i \leqq n.$$

Now B may be taken as the image under a Lipschitz mapping of a subset of Q. Also it follows immediately from 6.5 that (*) $H^k(B) \geqq \mu_n^k(B)$. By [6], this Lipschitz mapping may be extended to all of Q, with the same Lipschitz constant. Hence, in view of (*) without loss of generality, we may assume that

$$B = T(Q), \qquad T \text{ a Lipschitz mapping.}$$

Now let u be any point of Q, and g any element of G_n. Let
$J(u)$ be the Jacobian of T at u;
$J_g^*(u)$ be the Jacobian of gT at u;
$J_g(u)$ be the Jacobian of $I^k gT$ at u.
Now by ([7], 4.2), $J(u) = J_g^*(u)$. Also all of the above Jacobians exist L_k almost everywhere on Q. Hence there is a Borel subset E of Q such that
(1) T is univalent on E,
(2) All of the Jacobians exist and have positive, finite absolute values everywhere on E.

$$\mu_n^k(B - T(E)) = H^k(B - T(E)) = 0. \tag{3}$$

Now let u be any point in E. By (2) there is a k-plane π which is spanned by the k column vectors of the differential matrix of T at u. Select g in G_n such that $g(\pi) = R^k$. Then for this g,

$$J(u) = J_g^*(u) = J_g(u).$$

Thus for $\epsilon > 0$ there can be selected a sequence $g_1, g_2, \cdots$ in G_n such that the sets

(4) $$E_i^* = \{u \mid u \in E, 0 \leqq |J(u) - J_{gi}(u)| < \epsilon\}.$$

cover E. Let

$$E_1 = E_1^*, \qquad E_i = E_i^* - \bigcup_{j=1}^{i-1} E_j^*.$$

Now, from 6.7 if necessary a second sequence $g_1', g_2', \cdots$ can be selected from G_n, such that for u in E_i,

(5) $$0 \leqq |J(u) - J_{gi}(u)| < \epsilon,$$

and

(6) $$\mu_n^k(T(E_i)) \geqq \int_{R^k} N(x'', I^k g_i', T(E_i))dL_k.$$

Hence, from ([7], 4.5 Theorem), we have

$$\begin{aligned} H^k(B) &= \int_E J(u)dL_k = \sum_{i=1}^{\infty}\int_{E_i} J(u)dL_k \leqq \sum_{i=1}^{\infty}\int_{E_i} J_{g_i'}(u)dL_k + \epsilon \\ &= \sum_{i=1}^{\infty}\int_{R_k} N(x'', I^k g_i', T(E_i))dL_k + \epsilon \leqq \sum_{i=1}^{\infty} \mu_n^k(T(E_i)) + \epsilon = \mu_n^k(B) + \epsilon. \end{aligned}$$

Hence

$$H^k(B) \leqq \mu_n^k(B).$$

VII. The decomposition theorem.

7.1. LEMMA. *Let A be a closed (μ_n^k, k) unrectifiable set in R^n, such that $\mu_n^k(A) < +\infty$. Then $\mu_n^k(A) = 0$.*

Proof. For any g in G_n,

$$\begin{aligned} A = {} & [A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g))] \\ & \cup [\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g)]. \end{aligned}$$

By 3.15

$$\mu_n^k[A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n. By 6.4 and 6.6,

$$L_k[I^k g(A - (\overline{W}(\mu_n^k, A, g) \cup \overline{X}(\mu_n^k, A, g) \cup \overline{V}(A, g)))] = 0,$$

for σ_n almost every g in G_n. Likewise by 4.2 and 6.6 and 6.4, we have that

$$L_k[I^k g(X(\mu_n^k, A, g))] = 0,$$

for σ_n almost every g in G_n. Likewise by 5.1,

$$L_k[I^k g(\overline{W}(\mu_n^k, A, g))] = 0,$$

for every g in G_n. Likewise by 5.2 and 6.7, we have that

$$L_k[I^k g(\overline{V}(A, g))] = 0,$$

for σ_n almost every g in G_n. Hence, for σ_n almost every g in G_n,

$$L_k[I^k g(A)] = 0,$$

and therefore

$$\mu_n^k(A) = 0.$$

7.2. Theorem. *Let A be a Borel set in R^n such that $\mu_n^k(A) < +\infty$. Then $A = A_1 \cup A_2$, where A_1, A_2 are Borel sets. $A_1 \cap A_2 = \emptyset$. A_1 is countably k-rectifiable, and $\mu_n^k(A_2) = 0$.*

Proof. Let

$$\gamma = \text{l.u.b.}\ \mu_n^k(A'),$$

for A' a k-rectifiable subset of A. For each positive integer i, there is a k-rectifiable set $B_i \subset A$ such that

$$\mu_n^k(B_i) > \gamma - 1/i.$$

Then there is a Lipschitz transformation from a bounded set E_i in R^k onto B_i. By [6] this transformation can be extended to the whole of R^k with the same Lipschitz constant. If F_i is a bounded closed set in R^k containing E_i then the intersection of A and the image of F_i under this extended transformation is a Borel k-rectifiable subset of A containing B_i. Hence we may assume that B_i is a Borel set for each i. Set

$$A_1 = \bigcup_{i=1}^{\infty} B_i.$$

Then A_1 is a Borel and countably k-rectifiable set. The set $A_2 = A - A_1$ is Borel and (μ_n^k, k) unrectifiable, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. Since A_2 is the union of a countable number of closed (μ_n^k, k) unrectifiable sets and a set of μ_n^k measure 0, it follows from 7.1 that $\mu_n^k(A_2) = 0$.

7.3. Theorem. *If A is a Borel subset of R^n and $\mu_n^k(A) < +\infty$, then $\mu_n^k(A) = F_n^k(A)$, where F_n^k is the integralgeometric Favard k-measure of A, in R^n.* (*See* [2], 2.18, 5.11)

Proof. By 7.2, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$, A_1, A_2 Borel, A_1 is countably k-rectifiable, and $\mu_n^k(A_2) = 0$. Thus

$$F_n^k(A_2) = 0.$$

Also by 6.8 and ([2], 5.14 Theorem),

$$\mu_n^k(A_1) = H^k(A_1) = F_n^k(A_1).$$

7.4. THEOREM. *Let m, k and n be positive integers, $k < n < m$, and let A be a Borel subset of R^n, which may be thought of as a subspace of R^m. Then*

$$\mu_n^k(A) = \mu_m^k(A).$$

Proof. Under the above assumptions Federer has shown ([8], 7) that

$$F_m^k(A) = F_n^k(A).$$

To begin, we assume $\mu_n^k(A) < +\infty$. Then, by 6.8 and 7.2, $A = A_1 \cup A_2$, A_i Borel for $i = 1, 2$, $A_1 \cap A_2 = \emptyset$. Also A_1 is a countably k-rectifiable subset of both R^n and R^m. So

$$\mu_n^k(A_1) = H^k(A_1) = \mu_m^k(A_1), \quad \text{and} \quad \mu_n^k(A_2) = F_n^k(A_2) = 0.$$

Hence $F_m^k(A_2) = 0 = \mu_m^k(A_2)$. Therefore $\mu_n^k(A) = \mu_m^k(A)$. We now wish to show that if $\mu_n^k(A) = +\infty$, then $\mu_m^k(A) = +\infty$. To show this it suffices to show that if $\mu_m^k(A) < +\infty$, then $\mu_m^k(A) = \mu_n^k(A)$. But this follows from exactly the same type of argument as used above.

BIBLIOGRAPHY

1. E. J. Mickle, *On a decomposition theorem of Federer*, Trans. Amer. Math. Soc. vol. 92 (1959) pp. 322–335.

2. H. Federer, *The (ϕ, k) rectifiable subsets of n space*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 114–192.

3. E. J. Mickle and T. Radó, *Density theorems for outer measures in n space*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 433–439.

4. ———, *On density theorems for outer measures*, Interim technical report 3, The Ohio State University Research Foundation, 1958, p. 64.

5. A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 205–235.

6. E. J. Mickle, *On the extension of a transformation*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 160–164.

7. H. Federer, *Surface area*. II, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 438–456.

8. ———, *Dimension and measure*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 536–547.

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